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Time evolution of negative binomial optical field in diffusion channel*

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We find time evolution law of negative binomial optical field in diffusion channel. We reveal that by adjusting the diffusion parameter, photon number can controlled. Therefore, the diffusion process can be considered a quantum controlling scheme through photon addition.

Keywords: negative binomial optical field, time evolution, diffusion channel, integration within an ordered product (IWOP) of operators

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1. Introduction

In a recent paper [1] we have pointed out that an initial number state $|l\rangle\langle l|$ undergoing through a diffusion channel, described by the master equation [2-3]

$$\frac{d}{dt}\rho = -\kappa\left(a^{\dagger}a\rho + \rho aa^{\dagger} - a\rho a^{\dagger} - a^{\dagger}\rho a\right), \tag{1}$$

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would become a new photon optical field, named Laguerre-polynomial-weighted chaotic state, whose density operator is

$$\rho \equiv \lambda (1-\lambda)^{l} : L_{l}\left(\frac{-\lambda^{2} a^{\dagger} a}{1-\lambda}\right) e^{-\lambda a^{\dagger} a} : , \lambda=\frac{1}{1+\kappa t} \tag{2}$$

Here :: denotes normal ordering symbol, $|l\rangle = a^{\dagger n}|0\rangle/\sqrt{n!}$, L_l is the Laguerre polynomial. Experimentally, this new mixed state may be implemented, i.e., when a number state enters into the diffusion channel. Remarkablely, this state is characteristic of possessing photon number $Tr\left(a^{\dagger} a \rho\right) = l + \kappa t$ at time t, so we can control photon number by adjusting the diffusion parameter κ, this mechanism may provide application in quantum controlling.

To go a step further, our aim in this paper is to derive evolution law of a negative binomial state (NBS) in diffusion channel. Physically, when an atom absorbs some photons from a thermo light beam, then the corresponding photon field will be in a negative binomial state. We are thus challenged by the question: how an initial NBS evolves in a diffusion channel, what a final state it will be, and what is the photon number distribution in the final state. To our knowledge, such questions has not been touched in the literature before.

Our paper is arranged as follows. In Sec. 2, we convert the density operator of NBS into normally ordered form. In Sec. 3 based on the Kraus-operator-solution corresponding to the diffusion channel we find the evolution law of NBS in diffusion channel. Then in Sec. 4 we calculate the photon number distribution in the final state.

2. Normally ordered form of the density operator of negative binomial state

Corresponding to the negative-binomial formula

$$\sum_{m=0}^{\infty}\binom{m+n}{m}(-x)^{m}=(1+x)^{-n-1} \tag{3}$$

there exists negative-binomial state of quantum optical field [4]

$$\rho_{0}=\sum_{n=0}^{\infty} \frac{(n+s)!}{n! s!} \gamma^{s+1}(1-\gamma)^{n}|n\rangle\langle n|, 0<\gamma<1, \tag{4}$$

where $|n\rangle = a^{\dagger n}|0\rangle/\sqrt{n!}$ is the Fock state, $a^{\dagger}$ is the photon creation operator, $|0\rangle$ is the vacuum state in Fock space. NBS is intermediate between a pure thermal state and a pure coherent

state, and its nonclassical properties and algebraic characteristic have already been studied in detail in Refs. [11, 12]. The photon number average in this state is

$$Tr\left(\rho_0 a^\dagger a\right) = \frac{(s+1)(1-\gamma)}{\gamma} \tag{5}$$

Using $\left[a, a^\dagger\right] = 1$, $a^s|n\rangle = \sqrt{\frac{n!}{(n-s)!}}|n-s\rangle$, one can reform Eq. (4) as

$$\begin{aligned}
\rho_0 &= \frac{\gamma^{s+1}}{s!(1-\gamma)^s} a^s \sum_{n=0}^{\infty} (1-\gamma)^n |n\rangle\langle n| a^{\dagger s} \\
&= \frac{1}{s! n_c^s} a^s \rho_c a^{\dagger s}
\end{aligned} \tag{6}$$

where ρ_c denotes a chaotic field

$$\begin{aligned}
\rho_c &= \gamma \sum_{n=0}^{\infty} (1-\gamma)^n |n\rangle\langle n| = \gamma \sum_{n=0}^{\infty} \frac{(1-\gamma)^n}{n!} : a^{\dagger n} e^{-a^\dagger a} a^n := \gamma : e^{-\gamma a^\dagger a} : \\
&= \gamma e^{a^\dagger a \ln(1-\gamma)},
\end{aligned} \tag{7}$$

Eq. (6) tells us that when some photons are detected for a chaotic state, e.g. after detecting several photons, the chaotic light field exhibits negative-binomial distribution. One can further show $Tr\ _c = 1$, and

$$tr\left(\rho_c a^\dagger a\right) = \frac{1}{\gamma} - 1 = n_c \tag{8}$$

is the mean number of photons of chaotic light field, according to Bose-Einstein distribution, $n_c = \frac{1}{e^{\beta\omega\hbar}-1}$, here $\beta = \frac{1}{k_B T}$, k_B is the Boltzmann constant, ω is the frequency of chaotic light field. We can derive the normally ordered form of the density operator of NBS, let $\ln(1-\gamma) = f$, then $n_c = e^f/\left(1-e^f\right)$ and by introducing the coherent state representation $\int \frac{d^2z}{\pi}|z\rangle\langle z| = 1$, $|z\rangle = e^{\frac{-|z|^2}{2}} e^{za^\dagger}|0\rangle$, and employing the technique of integration within an ordered product (IWOP) of operators [5-6] we reform Eq. (6) as

$$\begin{aligned}
\rho_0 &= \frac{1}{s! n_c^s} a^s \rho_c a^{\dagger s} = \frac{1-e^f}{s! n_c^s} a^s e^{f a^\dagger a} a^{\dagger s} \\
&= \frac{1-e^f}{s! n_c^s} \int \frac{d^2z}{\pi} a^s e^{f a^\dagger a} |z\rangle\langle z| a^{\dagger s} \\
&= \frac{1-e^f}{s! n_c^s} \int \frac{d^2z}{\pi} e^{\frac{-|z|^2}{2}} a^s e^{f a^\dagger a} e^{z a^\dagger} e^{-f a^\dagger a} |0\rangle\langle z| z^{*s} \\
&= \frac{1-e^f}{s! n_c^s} \int \frac{d^2z}{\pi} e^{\frac{-|z|^2}{2}} a^s e^{z a^\dagger e^f} |0\rangle\langle z| z^{*s} \\
&= \frac{1-e^f}{s! n_c^s} \int \frac{d^2z}{\pi} \left(z e^f\right)^s z^{*s} : e^{-|z|^2 + z a^\dagger e^f + z^* a - a^\dagger a} :
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-e^f}{s!n_c^s} e^{fs} \colon \sum_{l=0}^{\infty} e^{\left(e^f-1\right)a^\dagger a} \frac{(n!)^2 \left(a^\dagger a e^f\right)^{n-l}}{l! \left[(n-l)!\right]^2} \colon \\
&= \left(1-e^f\right)^{s+1} \colon e^{\left(e^f-1\right)a^\dagger a} L_s\left(-a^\dagger a e^f\right) \colon \\
&= \gamma^{s+1} \colon e^{-\gamma a^\dagger a} L_s\left[(\gamma-1) a^\dagger a\right] \colon
\end{aligned} \tag{9}
$$

where we have used $|0\rangle\langle 0| =\colon e^{-a^\dagger a}\colon$, and the definition of Laguerre polynomials

$$
L_s(x) = \sum_{l=0}^{s} \frac{(-x)^l n!}{(l!)^2 (n-l)!} \tag{10}
$$

that (9) is quite different from (1), so they represent different optical field.

3. Evolution law of the negative binomial state in diffusion channel

Recall in Ref. [7] by using the entangled state representation and IWOP technique we have derived the infinite sum form of $\rho(t)$

$$
\begin{aligned}
\rho(t) &= \sum_{m,n=0}^{\infty} \sqrt{\frac{1}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t+1)^{m+n+1}}} a^{\dagger m} \left(\frac{1}{1+\kappa t}\right)^{a^\dagger a} \\
&\quad \times a^n \rho_0 a^{\dagger n} \left(\frac{1}{1+\kappa t}\right)^{a^\dagger a} a^m \sqrt{\frac{1}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t+1)^{m+n+1}}} \\
&= \sum_{m,n=0}^{\infty} M_{m,n} \rho_0 M_{m,n}^\dagger
\end{aligned} \tag{11}
$$

where

$$
M_{m,n} = \sqrt{\frac{1}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t+1)^{m+n+1}}} a^{\dagger m} \left(\frac{1}{1+\kappa t}\right)^{a^\dagger a} a^n \tag{12}
$$

satisfying $\sum_{m,n=0}^{\infty} M_{m,n}^\dagger M_{m,n} = 1$, which is trace conservative.

Now we examine time evolution of negative binomial optical field in diffusion channel. Substituting $\rho_0 = \frac{1}{s!n_c^s} a^s \rho_c a^{\dagger s}$ into (11) we have

$$
\begin{aligned}
\rho(t) &= \frac{\gamma^{s+1}}{s!(1-\gamma)^s} \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t+1)^{m+n+1}} \\
&\quad \times a^{\dagger m} \left(\frac{1}{1+\kappa t}\right)^{a^\dagger a} a^{n+s} e^{a^\dagger a \ln(1-\gamma)} a^{\dagger s+n} \left(\frac{1}{1+\kappa t}\right)^{a^\dagger a} a^m
\end{aligned} \tag{13}
$$

in which we first consider the summation over n, using

$$
\left(\frac{1}{1+\kappa t}\right)^{a^\dagger a} = e^{-a^\dagger a \ln(1+\kappa t)}, \quad e^{f a^\dagger a} a e^{-f a^\dagger a} = a e^{-f}, \tag{14}
$$

we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(\kappa t)^{n}}{(\kappa t+1)^{n}}e^{-a^{\dagger}a\ln(1+\kappa t)}a^{n+s}e^{a^{\dagger}a\ln(1-\gamma)}a^{\dagger s+n}e^{-a^{\dagger}a\ln(1+\kappa t)} \\ =\ &(1+\kappa t)^{2s}\sum_{n=0}^{\infty}\frac{(\kappa t)^{n}(1+\kappa t)^{n}}{n!}a^{n+s}e^{a^{\dagger}a\ln\left[(1-\gamma)/(1+\kappa t)^{2}\right]}a^{\dagger s+n}\end{aligned}\tag{15}$$

Note from Eq. (9) we have

$$a^{s}e^{fa^{\dagger}a}a^{\dagger s}=s!e^{fs}\colon e^{\left(e^{f}-1\right)a^{\dagger}a}L_{s}\left(-a^{\dagger}ae^{f}\right)\colon\tag{16}$$

it follows

$$\begin{aligned}&a^{n+s}e^{a^{\dagger}a[\ln(1-\gamma)-2\ln(1+\kappa t)]}a^{\dagger s+n} \\ =\ &(n+s)!e^{(n+s)\ln\left[(1-\gamma)/(1+\kappa t)^{2}\right]}\colon e^{\left[(1-\gamma)/(1+\kappa t)^{2}-1\right]a^{\dagger}a}L_{n+s}\left(-a^{\dagger}a\frac{1-\gamma}{(1+\kappa t)^{2}}\right)\colon\end{aligned}\tag{17}$$

Substituting (17) into (15) and multiplying $\frac{\gamma^{s+1}}{(1-\gamma)^{s}s!}$ we see

$$\begin{aligned}&\tfrac{\gamma^{s+1}(1+\kappa t)^{2s}}{(1-\gamma)^{s}}\textstyle\sum_{n=0}^{\infty}\tfrac{(\kappa t+1)^{n}(\kappa t)^{n}(n+s)!}{s!n!}e^{(n+s)\ln\left[(1-\gamma)/(1+\kappa t)^{2}\right]}\colon e^{\left[(1-\gamma)/(1+\kappa t)^{2}-1\right]a^{\dagger}a}L_{n+s}\left(-a^{\dagger}a\tfrac{(1-\gamma)}{(1+\kappa t)^{2}}\right)\colon \\ &=\gamma^{s+1}\textstyle\sum_{n=0}^{\infty}\tfrac{(n+s)!(\kappa t)^{n}(1-\gamma)^{n}}{s!n!(\kappa t+1)^{n}}\colon L_{n+s}\left(-a^{\dagger}a\tfrac{(1-\gamma)}{(1+\kappa t)^{2}}\right)e^{\left[(1-\gamma)/(1+\kappa t)^{2}-1\right]a^{\dagger}a}\colon\end{aligned}\tag{18}$$

Then we use the new generating function formula about the Laguerre polynomials [8].

$$\sum_{n=0}^{\infty}\frac{(n+s)!\,(-\lambda)^{n}}{n!s!}L_{n+s}(z)=(1+\lambda)^{-s-1}\,e^{\frac{\lambda z}{1+\lambda}}L_{s}\left(\frac{z}{1+\lambda}\right).\tag{19}$$

we obtain

$$\begin{aligned}(18)\ &=\left[\frac{\gamma(\kappa t+1)}{1+\kappa t\gamma}\right]^{s+1}\colon L_{s}\left(-a^{\dagger}a\frac{(1-\gamma)}{(1+\kappa t\gamma)(1+\kappa t)}\right)e^{a^{\dagger}a\frac{\kappa t(1-\gamma)^{2}}{(1+\kappa t\gamma)(1+\kappa t)^{2}}}e^{\left[(1-\gamma)/(1+\kappa t)^{2}-1\right]a^{\dagger}a} \\ &=\left[\frac{\gamma(\kappa t+1)}{1+\kappa t\gamma}\right]^{s+1}\colon L_{s}\left(-a^{\dagger}a\frac{(1-\gamma)}{(1+\kappa t\gamma)(1+\kappa t)}\right)e^{\left[-\frac{\gamma-1}{(t\kappa+1)(t\kappa\gamma+1)}-1\right]a^{\dagger}a}\colon\end{aligned}\tag{20}$$

For $\rho(t)$ in Eq. (13) It remains to perform summation over m, using the summation technique within normal ordering we have

$$\begin{aligned}\rho(t)\ &=\left[\frac{\gamma(\kappa t+1)}{1+\kappa t\gamma}\right]^{s+1}\sum_{m=0}^{\infty}\frac{(\kappa t)^{m}}{m!(\kappa t+1)^{m+1}} \\ &\times\colon a^{\dagger m}e^{\left[-\frac{\gamma-1}{(t\kappa+1)(t\kappa\gamma+1)}-1\right]a^{\dagger}a}L_{s}\left(-a^{\dagger}a\frac{(1-\gamma)}{(1+\kappa t\gamma)(1+\kappa t)}\right)a^{m}\colon \\ &=C\colon e^{Ea^{\dagger}a}L_{s}\left(a^{\dagger}aF\right)\colon\end{aligned}\tag{21}$$

where

$$E=-\frac{\gamma}{\kappa t\gamma+1},F=\frac{\gamma-1}{(1+\kappa t\gamma)(1+\kappa t)},F+E=\frac{-1}{1+\kappa t}\tag{22}$$

$$C=\frac{1}{1+\kappa t}\left[\frac{\gamma\left(\kappa t+1\right)}{1+\kappa t\gamma}\right]^{s+1} \tag{23}$$

Comparing $\rho\left(t\right)$ in (21) with ρ_0 in (9) we can see the big difference. Now we must check if $Tr\rho\left(t\right)=1$, in fact, using the coherent state's completeness relation $1=\int\frac{d^2z}{\pi}\left|z\right\rangle\left\langle z\right|$ and

$$\int_0^\infty e^{-bx}L_l\left(x\right)dx=\left(b-1\right)^l b^{-l-1}. \tag{24}$$

we do have

$$\begin{aligned}
Tr\rho\left(t\right) &= CTr\left[:e^{Ea^\dagger a}L_s\left(a^\dagger aF\right):\int\frac{d^2z}{\pi}\left|z\right\rangle\left\langle z\right|\right]\\
&= C\int\frac{d^2z}{\pi}e^{-\frac{\gamma}{t\kappa\gamma+1}\left|z\right|^2}L_s\left(-\frac{\left|z\right|^2\left(1-\gamma\right)}{\left(1+\kappa t\gamma\right)\left(1+\kappa t\right)}\right)\\
&= C\frac{\left(1+\kappa t\gamma\right)\left(1+\kappa t\right)}{\left(\gamma-1\right)}\int_0^\infty dr'e^{-\frac{\gamma\left(1+\kappa t\right)r'}{\left(\gamma-1\right)}}L_s\left(r'\right)\\
&= C\frac{\left(1+\kappa t\gamma\right)\left(1+\kappa t\right)}{\gamma-1}\left(\frac{\gamma\kappa t+1}{\gamma-1}\right)^s\left[\frac{\gamma-1}{\gamma\left(1+\kappa t\right)}\right]^{s+1}=1
\end{aligned} \tag{25}$$

4. Photon number average in the final state

Now we evalute photon number average in the final state, using (21), (10) and (22)-(24) we have

$$\begin{aligned}
Tr\left[\rho\left(t\right)a^\dagger a\right] &= C\int\frac{d^2z}{\pi}\left\langle z\right|a^\dagger a:e^{Ea^\dagger a}L_s\left(a^\dagger aF\right):\left|z\right\rangle \qquad (26)\\
&= C\int\frac{d^2z}{\pi}z^*\left\langle z\right|\left\{\left[a,:e^{Ea^\dagger a}L_s\left(a^\dagger aF\right):\right]+:e^{Ea^\dagger a}L_s\left(a^\dagger aF\right):a\right\}\left|z\right\rangle\\
&= C\int\frac{d^2z}{\pi}\left\langle z\right|z^*\left\{\frac{\partial}{\partial a^\dagger}:e^{Ea^\dagger a}L_s\left(a^\dagger aF\right):+\left|z\right|^2e^{E\left|z\right|^2}L_s\left(\left|z\right|^2F\right)\right\}\left|z\right\rangle\\
&= C\int\frac{d^2z}{\pi}e^{E\left|z\right|^2}\sum_{l=0}^s\frac{l\left|z\right|^{2l}\left(-F\right)^l s!}{\left(l!\right)^2\left(s-l\right)!}+C\int\frac{d^2z}{\pi}\left(1+E\right)\left|z\right|^2e^{E\left|z\right|^2}L_s\left(\left|z\right|^2F\right)\\
&= C\sum_{l=0}^s\frac{\left(-F\right)^l s!}{\left(l!\right)^2\left(s-l\right)!}\left[l\int\frac{d^2z}{\pi}e^{E\left|z\right|^2}\left|z\right|^{2l}+\left(1+E\right)\int\frac{d^2z}{\pi}\left|z\right|^{2\left(l+1\right)}e^{E\left|z\right|^2}\right]\\
&= C\sum_{l=0}^s\frac{\left(-F\right)^l s!}{\left(l!\right)^2\left(s-l\right)!}\left[l\frac{l!}{\left(-E\right)^{l+1}}+\left(1+E\right)\frac{\left(l+1\right)!}{\left(-E\right)^{l+2}}\right]\\
&= tk+\frac{\left(s+1\right)\left(1-\gamma\right)}{\gamma}
\end{aligned}$$

Comparing with Eq. (5) we see that after passing through a diffusion channel, the photon average of a NBS varies from $\frac{(s+1)(1-\gamma)}{\gamma}$ to $tk+\frac{(s+1)(1-\gamma)}{\gamma}$,

$$Tr\left[\rho(t)a^\dagger a\right]=tk+\frac{\left(s+1\right)\left(1-\gamma\right)}{\gamma}=tk+Tr\left(\rho_0 a^\dagger a\right) \tag{27}$$

This result is encourageous, since by adjusting the diffusion parameter κ, we can control photon number, when κ is small, it slightly increases by an amount κt. Therefore, this diffusion process for NBS can be considered a quantum controlling scheme through photon addition.

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